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On the Continuity of the Set-Valued Exponential Metric Projection

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In this note we consider the Chebyshevian approximation problem for a compact real interval and the class of exponential sums. We give necessary and sufficient conditions for the continuity of the associated set-valued metric projection.

Exponential sums are functions which can be written

$$h(x) = \sum_{i=1}^{l} p_i(x)\, e t_i^x,$$

where the t_i are real and distinct and the p_i are polynomials with real coefficients. The expression

$$k(h) := \sum_{i=1}^{l} (\partial p_i + 1),$$

will be referred to as the degree of the exponential sum h. Here ∂p denotes the degree of p (if $p \equiv 0$, ∂p is -1). The set of all exponential sums having a degree less than or equal to a natural number n is denoted by V_n. Let $[a, b]$ be a closed real interval and $C[a, b]$ the space of all real-valued functions defined and continuous on $[a, b]$, with the Chebyshev norm

$$\|f\| := \sup\{|f(x)|, x \in [a, b]\}.$$

An element $h^* \in V_n$ is called a minimal solution or best approximant of f with respect to V_n iff

$$\|f - h^*\| = \inf\{\|f - h\|, h \in V_n\}.$$

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The set of all minimal solutions is called the metric projection of $C[a, b]$ onto V_n at the point f, and is denoted by $P_n(f)$. This set is not empty and in general has more than one element [1].

There are many ways to define and characterize the continuity of such a set-valued function, see, [8–10]. For the purpose of this work, the following definition seems to be most appropriate, cf. [10]. The topology in $C[a, b]$ is understood to be that defined by the Chebyshev norm.

DEFINITION 1. The metric projection is called upper semicontinuous at the point $f \in C[a, b]$ iff for every open set $U \subset V_n$ with $P_n(f) \subset U$, the set $\{g \in C[a, b], P_n(g) \subset U\}$ is open. The metric projection is called lower semicontinuous at the point $f \in C[a, b]$ iff for every open set with $U \cap P_n(f) \neq \varnothing$, the set $\{g \in C[a, b], P_n(g) \cap U \neq \varnothing\}$ is open. The metric projection is called continuous at the point $f \in C[a, b]$ iff it is both upper and lower semicontinuous at f.

We need some results on the topological nature of the exponential sums [5].

LEMMA 1. *The set*

$$V_{n,K} := \{h \in V_n, \|h\| \leqslant K\},$$

where $0 < K < \infty$, contains a sequence which converges uniformly on every compact subinterval of (a, b) to an element of $V_{n,K}$.

LEMMA 2. *Let $\{h_m\} \subset V_n$ be a sequence which is bounded in $[a, b]$ and converges uniformly on every compact subinterval of (a, b). Let h be the limiting function and let $k(h_m) \leqslant k(h)$. Then $\{h_m\}$ converges uniformly on $[a, b]$ to h.*

A simple consequence of Lemma 1 is the following lemma.

LEMMA 3. *Let $f \in C[a, b]$, $f_m \in C[a, b]$ $(m = 1, 2,...)$ satisfy*

$$\lim \|f_m - f\| = 0,$$

and let $h_m \in P_n(f_m)$. Then the sequence $\{h_m\}$ contains a subsequence which converges uniformly on every compact subinterval of (a, b) to a best approximant of f.

The proof of the following lemma uses an idea of Braess [2].

LEMMA 4. *The set of all functions of $C[a, b]$ which have a unique best approximant is dense in $C[a, b]$.*

Proof. Let $g \in C[a, b]$ and U be a neighborhood of g. By the Stone–Weierstrass theorem there is a continuously differentiable function f in U.

Let $h_1 \in P_n(f)$. We will show that for any δ, $0 < \delta < 1$, the function $f_\delta := h_1 + (1 - \delta)(f - h_1)$ has h_1 as its unique best approximant, which by $\|f_\delta - f\| = \delta \|f - h_1\|$ implies the statement of the lemma. Assume that there is a $h^* \in V_n$, which is a better approximant for f_δ than h_1. Then, with $\eta := \|f - h_1\|$ we would have

$$(1 - \delta)\,\eta = \|f_\delta - h_1\| > \|f_\delta - h^*\| \geqslant \|f - h^*\| - \|f - f_\delta\|,$$

which contradicts the minimal property of h_1 for f. Hence $h_1 \in P_n(f_\delta)$. Assume that there is another element h_2 of $P_n(f_\delta)$. It is easily seen that h_2 is also in $P_n(f)$. There exists an alternant $A = \{x_1, \ldots, x_p\}$ of $f - h_2$ with $p \geqslant n + 2$, cf. [1]. From the representation $f - h_2 = \delta(f - h_1) + (f_\delta - h_2)$ and recalling that $\|f - h_1\| = \eta$, $\|f_\delta - h_2\| = (1 - \delta)\,\eta$, $0 < \delta < 1$, we have that for all $x \in A$,

$$f_\delta(x) - h_2(x) = (1 - \delta)(f(x) - h_1(x)) = f_\delta(x) - h_1(x).$$

Thus $h_1(x) = h_2(x)$ for all $x \in A$. At the points of $A \cap (a, b)$ we have in addition $h_1'(x) = h_2'(x)$. Therefore the difference $h_1 - h_2$ has at least $2n$ zeros, counting multiplicities. Since its degree is at most $2n$ it follows from [3, p. 167; 4] that $h_1 - h_2 \equiv 0$.

This result leads to the following theorem.

THEOREM 1. *Let $f \in C[a, b]$ and let $P_n(f)$ contain more than one element, then the metric projection is not continuous at f.*

Proof. We show that P_n is not lower semicontinuous at f. By Lemmas 3 and 4 there is a sequence $\{f_m\} \subset C[a, b[$ with the following properties:

(i) $\lim \|f_m - f\| = 0$,

(ii) h_m is the only element in $P_n(f_m)$, and

(iii) $\lim |h_m(x) - h_1(x)| = 0$ for all $x \in (a, b)$, where h_1 is some element in $P_n(f)$. Let $h_2 \in P_n(f)$, $h_1 \neq h_2$, and let $U \subset V_n$ be an open set such that $h_2 \in U$, $h_1 \notin U$. Then for m sufficiently large we have $h_m \notin U$, i.e., $P_n(f_m) \cap U = \varnothing$. Thus the metric projection is not lower semicontinuous and, therefore, not continuous at f.

As is the case with other approximating families (rational and varisolvent functions), continuity is closely related to the question whether the best approximant has maximum degree. A function $f \in C[a, b]$ is said to be normal iff $P_n(f) \cap V_{n-1} = \varnothing$.

THEOREM 2. *The metric projection is upper semicontinuous at any normal point of $C[a, b]$.*

Proof. Let $\{h_m\} \subset V_n$ be a minimizing sequence for f, i.e., $\lim \|f - h_m\| = \inf\{\|f - h\|, h \in V_n\}$. From Lemmas 1 and 2 it follows that $\{h_m\}$ has an accumulation point in V_n. Therefore, by a result of Singer [7], the metric projection is upper semicontinuous at f.

Let $g \in C[a, b]$, $g \not\equiv 0$, then the maximal number of elements of all alternants of g will be called the length of alternation of g and will be denoted by $\operatorname{alt}(g)$.

The following theorem shows that under an additional assumption normality is also a necessary condition. Here $k(h)$ is the above defined degree of the exponential sum h.

THEOREM 3. *Let $f \in C[a, b] - V_n$ be not normal and let $h^* \in P_n(f) \cap V_{n-1}$. If one of the following conditions is satisfied, then the metric projection is not upper semicontinuous at f.*

(1) *There is an alternant of $f - h^*$ containing at least $n + k(h^*) + 1$ points.*

(2) *There is a neighborhood U of f such that for all $g \in U$ it is true that if a best approximant of g has degree at least $k(h^*) + 1$ then the length of alternation of the associated error function is at least*

$$\max\{\operatorname{alt}(f - h), h \in P_n(f)\} + 1.$$

Proof. If (1) holds, then the minimal solution of f is unique [1] and the assertion follows from [6, Definition 1 and Satz 4]. Now let (2) hold. We consider the sequences $\{f_m\} \subset C[a, b]$ and $\{h_m\} \subset V_n$, which are constructed in [6, Lemma 8]. These sequences have the following properties:

(i) $h_m \subset V_{k(h^*)+1}$,

(ii) $\operatorname{alt}(f_m - h_m) = \operatorname{alt}(f - h^*) + 1$,

(iii) $\|f_m - h_m\| = \|f - h\|$,

(iv) $\lim \|f_m - f\| = 0$,

(v) $\{h_m\}$ converges pointwise to a discontinuous function.

Without loss of generality we may assume $f_m \in U$. If $P_n(f_m) \cap V_{k(h^*)} \neq \varnothing$ define $\hat{h}_m := h_m$, otherwise select an arbitrary element $\hat{h}_m$ of $P_n(f_m)$. In both cases we have $k(\hat{h}_m) \geqslant k(h^*) + 1$. This implies, by hypothesis and the fact $\lim \|f_m - f\| = 0$ that the set $\{h_m, m \in N\}$ has no accumulation point and, therefore, is closed. Furthermore, for m sufficiently large, $h_m \notin P_n(f)$. Since $P_n(f)$ is closed, there exist open subsets U_1, U_2 of V_n such that for m sufficiently large, $h_m \in U_1$, $P_n(f) \subset U_2$, $U_1 \cap U_2 = \varnothing$. Therefore the metric projection is not upper semicontinuous at f.

THEOREM 4. *Let $f \in C[a, b]$ be normal and have a unique best approximant. Then the metric projection is continuous at f.*

Proof. By Theorem 2 the metric projection is upper semicontinuous, and by Lemmas 2 and 3 it is also lower semicontinuous at f.

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